

The binary Goldbach problem with arithmetic weights attached to one of the variables

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1 Introduction and statement of the results.

Suppose that N is a sufficiently large integer and denote

$$J(n) = \sum_{p_1+p_2=n} \log p_1 \log p_2.$$

(From this place the letter p , with or without subscripts, is reserved for primes.)

It is expected that if n is a large even integer then $J(n) \sim c_0 \lambda(n)n$, where

$$\lambda(k) = \prod_{\substack{p|k \\ p>2}} \frac{p-1}{p-2}, \quad c_0 = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right). \quad (1)$$

This conjecture has not been proved so far, but using the Hardy–Littlewood circle method and Vinogradov’s method for estimating exponential sums over primes (see, for example, Vaughan [11], Ch. 2), one can find that

$$\sum_{\substack{n \leq N \\ 2|n}} |J(n) - c_0 \lambda(n)n| \ll N^2 \mathcal{L}^{-A}, \quad (2)$$

where $A > 0$ is an arbitrarily large constant and $\mathcal{L} = \log N$.

Let $r(k)$ be the number of solutions of the equation $x_1^2 + x_2^2 = k$ in integers x_1, x_2 . One of the classical problems in prime number theory is the Hardy–Littlewood problem concerning the representation of large integers as a sum of two squares and a prime. It was solved by Linnik (see [7]) and related problems have been studied by Linnik, Hooley and other mathematicians. For

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more information we refer the reader to Hooley's book [5], Ch.5. In particular, one can show that

$$\sum_{p \leq N} r(p-1) = \pi N \mathcal{L}^{-1} \prod_{p > 2} \left(1 + \frac{\chi(p)}{p(p-1)}\right) + O(N \mathcal{L}^{-1-\theta_0} (\log \mathcal{L})^5), \quad (3)$$

where $\chi(k)$ is the non-principal character modulo 4 and

$$\theta_0 = \frac{1}{2} - \frac{1}{4}e \log 2 = 0.0029 \dots \quad (4)$$

Let $\tau(k)$ be the number of positive divisors of k . Linnik [7] (see also Halberstam and Richert [4], Ch. 3.5.) solved the Titchmarsh divisor problem and proved that

$$\sum_{p \leq N} \tau(p-1) = c_0 N + O(N \mathcal{L}^{-1} \log \mathcal{L}), \quad c_0 = \prod_p \left(1 + \frac{1}{p(p-1)}\right). \quad (5)$$

We note that sharper versions of (3) and (5) are known at present (see Bredihin [2], Bombieri, Friedlander and Iwaniec [1]) and Fouvry [3].

In this paper we state two theorems which are, in some sense, combinations of (2), (3) and respectively (2), (5). Denote

$$\mathcal{R}(n) = \sum_{p_1+p_2=n} r(p_1-1) \log p_1 \log p_2. \quad (6)$$

After certain formal calculations one may conjecture that for any sufficiently large even n the quantity $\mathcal{R}(n)$ is asymptotically equal to

$$\mathcal{M}_{\mathcal{R}}(n) = \pi c_0 n \prod_{p|n-1} \left(1 - \frac{\chi(p)}{p}\right) \prod_{\substack{p|n \\ p>2}} \left(1 + \frac{p + \chi(p)}{p(p-2)}\right) \prod_{p \nmid n(n-1)} \left(1 + \frac{2\chi(p)}{p(p-2)}\right). \quad (7)$$

Our first result is the following:

Theorem 1. *Suppose that θ_0 is the constant defined by (4). Then we have*

$$\sum_{\substack{n \leq N \\ 2|n}} |\mathcal{R}(n) - \mathcal{M}_{\mathcal{R}}(n)| \ll N^2 \mathcal{L}^{-\theta_0} (\log \mathcal{L})^6. \quad (8)$$

It is clear that $n (\log \log(10n))^{-2} \ll \mathcal{M}_{\mathcal{R}}(n) \ll n (\log \log(10n))^2$. Also, from (8) it follows that for any positive constant $\theta < \theta_0$ the number of even $n \leq N$ for which $|\mathcal{R}(n) - \mathcal{M}_{\mathcal{R}}(n)| > N \mathcal{L}^{-\theta}$ is $O(N \mathcal{L}^{-(\theta_0-\theta)} (\log \mathcal{L})^6)$. So, in other words, $\mathcal{R}(n)$ is close to $\mathcal{M}_{\mathcal{R}}(n)$ for almost all even n .

Theorem 1 is related to a recent result of K. Matomäki [8]. It is shown in [8] that the number of integers $n \leq N$ satisfying $n \equiv 0$ or $4 \pmod{6}$

and that cannot be represented as a sum of two primes, one of which of the form $k^2 + l^2 + 1$, is $O(N\mathcal{L}^{-A})$, where A is an arbitrarily large constant. So Matomäki's estimate for the cardinality of this exceptional set is stronger than ours, but her method does not provide so sharp information about the number of such representations.

Our second result is concerning the quantity

$$\mathcal{T}(n) = \sum_{p_1+p_2=n} \tau(p_1-1) \log p_1 \log p_2.$$

Again, after certain formal calculations, one may conclude that $\mathcal{T}(n)$ should be asymptotically equal to

$$\mathcal{M}_{\mathcal{T}}(n) = c_0 n \log n \prod_{p|n-1} \left(1 - \frac{1}{p}\right) \prod_{\substack{p|n \\ p>2}} \left(1 + \frac{p+1}{p(p-2)}\right) \prod_{p \nmid n(n-1)} \left(1 + \frac{2}{p(p-2)}\right).$$

We can establish:

Theorem 2. *The following estimate holds*

$$\sum_{\substack{n \leq N \\ 2|n}} |\mathcal{T}(n) - \mathcal{M}_{\mathcal{T}}(n)| \ll N^2 (\log \mathcal{L})^3.$$

We note that $n \log n (\log \log(10n))^{-2} \ll \mathcal{M}_{\mathcal{T}}(n) \ll n \log n (\log \log(10n))^2$, so the quantity $\mathcal{T}(n)$ is close to $\mathcal{M}_{\mathcal{T}}(n)$ for almost all even n .

We prove only Theorem 1. The proof of Theorem 2 is similar and simpler.

2 Some lemmas.

Suppose that $n \leq N$ and let k and l be integers with $(k, l) = 1$ (as usual, (k, l) stands for the greatest common factor of k and l). Let \mathcal{I} be the set of all subintervals of the interval $[1, N]$ and let $I \in \mathcal{I}$. We denote

$$J_{k,l}(n; I) = \sum_{\substack{p_1+p_2=n \\ p_1 \equiv l \pmod{k} \\ p_1 \in I}} \log p_1 \log p_2, \quad J_{k,l}(n) = J_{k,l}(n; [1, N]); \quad (9)$$

$$\mathfrak{S}_{k,l}(n) = \begin{cases} c_0 \lambda(nk) & \text{if } (k, n-l) = 1 \text{ and } 2 \mid n, \\ 0 & \text{otherwise;} \end{cases} \quad (10)$$

$$\Phi(n; I) = \sum_{\substack{m_1+m_2=n \\ m_1 \in I}} 1. \quad (11)$$

Our first lemma states that the expected formula for $J_{k,l}(n; I)$ is true on average with respect to $k \leq \sqrt{N}\mathcal{L}^{-B}$ and $n \leq N$ and uniformly for l and I . More precisely, we have

Lemma 1. *For any constant $A > 0$ there exist $B = B(A) > 0$ such that*

$$\sum_{k \leq \sqrt{N}\mathcal{L}^{-B}} \max_{(l,k)=1} \max_{I \in \mathcal{I}} \sum_{n \leq N} \left| J_{k,l}(n; I) - \frac{\mathfrak{S}_{k,l}(n)}{\varphi(k)} \Phi(n; I) \right| \ll N^2 \mathcal{L}^{-A}.$$

This lemma is very similar to results of Mikawa [9] and Laporta [6]. These authors study the equation $p_1 - p_2 = n$ and without the condition $p_1 \in I$. However inspecting the arguments presented in [6], the reader will readily see that the proof of Lemma 1 can be obtained in the same manner.

The next lemma is an immediate consequence from a classical sieve theory result (see [4], Ch. 2, Th. 2.4).

Lemma 2. *Suppose that h is an integer such that $1 \leq |h| \leq N$. Then the number of solutions of the equation $p_1 - p_2 = h$ in primes $p_1, p_2 \leq N$ is $O(N\mathcal{L}^{-2} \log \mathcal{L})$, where the constant in the Landau symbol is absolute.*

The next two lemmas are due to C. Hooley and play an essential role in the proof of (3), as well as in the solutions of other related problems.

Lemma 3. *Suppose that $\omega > 0$ is a constant and let $F_\omega(N)$ be the number of primes $p \leq N$ such that $p-1$ has a divisor lying between $\sqrt{N}\mathcal{L}^{-\omega}$ and $\sqrt{N}\mathcal{L}^\omega$. Then we have*

$$F_\omega(N) \ll N\mathcal{L}^{-1-2\theta_0} (\log \mathcal{L})^3,$$

where θ_0 is defined by (4) and where the constant in the Vinogradov symbol depends only on ω .

Lemma 4. *Suppose that $\omega > 0$ is a constant. Then we have*

$$\sum_{p \leq N} \left| \sum_{\substack{d|p-1 \\ \sqrt{N}\mathcal{L}^{-\omega} < d < \sqrt{N}\mathcal{L}^\omega}} \chi(d) \right|^2 \ll N\mathcal{L}^{-1} (\log \mathcal{L})^7,$$

where the constant in the Vinogradov symbol depends only on ω .

The proofs of very similar results (with $\omega = 48$ and with the condition $d \mid N - p$ rather than $d \mid p - 1$) are available in [5], Ch.5 and the reader will easily see that the method used there yields also the validity of Lemmas 3 and 4.

3 Proof of Theorem 1.

3.1 Beginning.

Denote by \mathcal{E} the sum on the left-hand side of (8) and put

$$D = \sqrt{N}\mathcal{L}^{-1-B(1)}, \quad (12)$$

where $B(A)$ is specified in Lemma 1. Using (6) and the well-known identity $r(m) = 4 \sum_{d|m} \chi(d)$ we find

$$\mathcal{R}(n) = 4 \sum_{p_1+p_2=n} \left(\sum_{\substack{d|p_1-1}} \chi(d) \right) \log p_1 \log p_2 = 4 (S_1(n) + S_2(n) + S_3(n)), \quad (13)$$

where

$$S_1(n) = \sum_{p_1+p_2=n} \left(\sum_{\substack{d|p_1-1 \\ d \leq D}} \chi(d) \right) \log p_1 \log p_2 \quad (14)$$

$$S_2(n) = \sum_{p_1+p_2=n} \left(\sum_{\substack{d|p_1-1 \\ D < d < N/D}} \chi(d) \right) \log p_1 \log p_2 \quad (15)$$

$$S_3(n) = \sum_{p_1+p_2=n} \left(\sum_{\substack{d|p_1-1 \\ d \geq N/D}} \chi(d) \right) \log p_1 \log p_2 \quad (16)$$

Therefore from (8) and (13) it follows

$$\mathcal{E} \ll \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3, \quad (17)$$

where

$$\mathcal{E}_1 = \sum_{\substack{n \leq N \\ 2|n}} |4S_1(n) - \mathcal{M}_{\mathcal{R}}(n)|; \quad \mathcal{E}_j = \sum_{\substack{n \leq N \\ 2|n}} |S_j(n)|, \quad j = 2, 3. \quad (18)$$

3.2 The estimation of \mathcal{E}_1 .

Using (9), (11), (14) and bearing in mind Lemma 1 we find

$$S_1(n) = \sum_{d \leq D} \chi(d) J_{d,1}(n) = (n-1)S'_1(n) + S_1^*(n),$$

where

$$S'_1(n) = \sum_{d \leq D} \chi(d) \frac{\mathfrak{S}_{d,1}(n)}{\varphi(d)}, \quad (19)$$

$$S_1^*(n) = \sum_{d \leq D} \chi(d) \left(J_{d,1}(n) - (n-1) \frac{\mathfrak{S}_{d,1}(n)}{\varphi(d)} \right). \quad (20)$$

Hence

$$\mathcal{E}_1 \ll \mathcal{E}'_1 + \mathcal{E}_1^*, \quad (21)$$

where

$$\mathcal{E}'_1 = \sum_{\substack{n \leq N \\ 2 \nmid n}} |4(n-1)S'_1(n) - \mathcal{M}_{\mathcal{R}}(n)|, \quad \mathcal{E}_1^* = \sum_{\substack{n \leq N \\ 2 \nmid n}} |S_1^*(n)|. \quad (22)$$

By (12), (20), (22) and Lemma 1 it follows that

$$\mathcal{E}_1^* \ll N^2 \mathcal{L}^{-1}. \quad (23)$$

Consider \mathcal{E}'_1 . From (1), (10) and (19) we find

$$S'_1(n) = c_0 \sum_{\substack{d \leq D \\ (d, n-1)=1}} \frac{\chi(d)}{\varphi(d)} \lambda(nd) = c_0 \lambda(n) \sum_{\substack{d \leq D \\ (d, n-1)=1}} f_n(d), \quad (24)$$

where

$$f_n(d) = \frac{\chi(d)}{\varphi(d)} \frac{\lambda(d)}{\lambda((n, d))}. \quad (25)$$

Obviously the function $f_n(d)$ is multiplicative with respect to d and

$$f_n(d) \ll d^{-1} (\log \log(10d))^2 \quad (26)$$

uniformly with respect to n . To evaluate the sum in right-hand side of (24) we consider the function

$$F_n(s) = \sum_{\substack{d=1 \\ (d, n-1)=1}}^{\infty} f_n(d) d^{-s}.$$

It is analytic in the half-plane $\operatorname{Re}(s) > 0$ and we may represent it as an Euler product:

$$F_n(s) = \prod_{p \nmid n-1} T_n(p, s), \quad T_n(p, s) = 1 + \sum_{l=1}^{\infty} f_n(p^l) p^{-ls}.$$

From (1) and (25) we easily find

$$f_n(p^l) = \begin{cases} \chi(p)^l p^{1-l} (p-1)^{-1} & \text{if } p \mid n, \\ \chi(p)^l p^{1-l} (p-2)^{-1} & \text{if } p \nmid n; \end{cases}$$

and respectively

$$T_n(p, s) = \left(1 - \frac{\chi(p)}{p^{s+1}}\right)^{-1} T_n^*(p, s),$$

where

$$T_n^*(p, s) = \begin{cases} 1 + \chi(p)p^{-s-1}(p-1)^{-1} & \text{if } p \mid n, \\ 1 + 2\chi(p)p^{-s-1}(p-2)^{-1} & \text{if } p \nmid n. \end{cases}$$

Therefore

$$F_n(s) = L(s+1, \chi)H_n(s) \quad (27)$$

where $L(s, \chi)$ is the Dirichlet L -function corresponding to the character χ and

$$H_n(s) = \prod_{p \mid n-1} \left(1 - \frac{\chi(p)}{p^{s+1}}\right) \prod_{p \mid n} \left(1 + \frac{\chi(p)}{p^{s+1}(p-1)}\right) \prod_{p \nmid n(n-1)} \left(1 + \frac{2\chi(p)}{p^{s+1}(p-2)}\right). \quad (28)$$

From (27), (28) we see that $F_n(s)$ has an analytic continuation to the half-plane $\operatorname{Re}(s) > -1$. It is clear that $H_n(s) \ll n^\varepsilon$ for $|\operatorname{Re}(s)| \geq -1/2$ (here and later ε is an arbitrarily small positive number). Also, it is well-known that in the same region we have $L(s+1, \chi) \ll 1 + |\operatorname{Im}(s)|^{1/6}$. Hence

$$F_n(s) \ll N^\varepsilon T^{1/6} \quad \text{if} \quad \operatorname{Re}(s) \geq -1/2, \quad |\operatorname{Im}(s)| \leq T \quad (29)$$

for any $T > 1$. We apply Perron's formula (see, for example [10], Ch. II.2) to find

$$\sum_{\substack{d \leq D \\ (d, n-1)=1}} f_n(d) = \frac{1}{2\pi i} \int_{\varkappa-iT}^{\varkappa+iT} F_n(s) \frac{D^s}{s} ds + O\left(\sum_{d=1}^{\infty} \frac{D^\varkappa |f_n(d)|}{d^\varkappa (1 + T |\log \frac{D}{d}|)}\right) \quad (30)$$

with $\varkappa = 1/10$ and $T = N^{3/4}$. Using (12) and (26) one can easily verify that the remainder term in (30) is $O(N^{-1/20})$. To evaluate the integral in (30) we apply Cauchy's theorem. The residue of the integrand at $s = 0$ equals

$$F_n(0) = \frac{\pi}{4} \prod_{p \mid n-1} \left(1 - \frac{\chi(p)}{p}\right) \prod_{p \mid n} \left(1 + \frac{\chi(p)}{p(p-1)}\right) \prod_{p \nmid n(n-1)} \left(1 + \frac{2\chi(p)}{p(p-2)}\right). \quad (31)$$

Hence the main term in the right-hand side of (30) is equal to

$$F_n(0) + \frac{1}{2\pi i} \left(\int_{\varkappa-iT}^{-1/2-iT} + \int_{-1/2-iT}^{-1/2+iT} + \int_{-1/2+iT}^{\varkappa+iT} \right) F_n(s) \frac{D^s}{s} ds. \quad (32)$$

Using (29) one can easily find that the contribution of the integrals in (32) is $O(N^{-1/20})$. Therefore

$$\sum_{\substack{d \leq D \\ (d, n-1)=1}} f_n(d) = F_n(0) + O(N^{-1/20}). \quad (33)$$

From (1), (7), (22), (24), (31) and (33) it follows that

$$\mathcal{E}'_1 \ll N^2 \mathcal{L}^{-1}.$$

Hence, using (21) and (23) we get

$$\mathcal{E}_1 \ll N^2 \mathcal{L}^{-1}. \quad (34)$$

3.3 The estimation of \mathcal{E}_2 .

Clearly, from (18) and Cauchy's inequality it follows that

$$\mathcal{E}_2 \ll N^{1/2} \left(\sum_{n \leq N} |S_2(n)|^2 \right)^{1/2} = N^{1/2} (\mathcal{E}'_2)^{1/2}, \quad (35)$$

say. Using (15) we find

$$\begin{aligned} \mathcal{E}'_2 &= \sum_{n \leq N} \sum_{D < d, t < N/D} \chi(d) \chi(t) \sum_{\substack{p_1 + p_2 = n \\ p_1 \equiv 1 \pmod{d}}} \log p_1 \log p_2 \sum_{\substack{p_3 + p_4 = n \\ p_3 \equiv 1 \pmod{t}}} \log p_3 \log p_4 \\ &= \sum_{p_1 + p_2 = p_3 + p_4 \leq N} \log p_1 \log p_2 \log p_3 \log p_4 \sum_{\substack{D < d, t < N/D \\ d|p_1-1, t|p_3-1}} \chi(d) \chi(t) \\ &\ll \mathcal{L}^4 \mathcal{E}_2'' + N^{2+\varepsilon}, \end{aligned} \quad (36)$$

where

$$\mathcal{E}_2'' = \sum_{\substack{p_1 + p_2 = p_3 + p_4 \\ p_1, p_2, p_3, p_4 \leq N \\ p_1 \neq p_3}} \left| \sum_{\substack{D < d < N/D \\ d|p_1-1}} \chi(d) \right| \left| \sum_{\substack{D < t < N/D \\ t|p_3-1}} \chi(t) \right|.$$

Denote by \mathcal{F} the set of primes $p \leq N$ such that $p-1$ has a divisor lying between D and N/D . Using the inequality $uv \leq u^2 + v^2$ and taking into

account the symmetry with respect to d and t we get

$$\begin{aligned}
\mathcal{E}_2'' &\ll \sum_{\substack{p_1+p_2=p_3+p_4 \\ p_1, p_2, p_4 \leq N \\ p_1 \neq p_3, p_3 \in \mathcal{F}}} \left| \sum_{\substack{D < d < N/D \\ d|p_1-1}} \chi(d) \right|^2 \\
&= \sum_{p_1 \leq N} \left| \sum_{\substack{D < d < N/D \\ d|p_1-1}} \chi(d) \right|^2 \sum_{\substack{p_3 \in \mathcal{F} \\ p_3 \neq p_1}} \sum_{\substack{p_2, p_4 \leq N \\ p_4 - p_2 = p_1 - p_3}} 1. \tag{37}
\end{aligned}$$

Applying Lemmas 2 and 3 we find

$$\sum_{\substack{p_3 \in \mathcal{F} \\ p_3 \neq p_1}} \sum_{\substack{p_2, p_4 \leq N \\ p_4 - p_2 = p_1 - p_3}} 1 \ll N \mathcal{L}^{-2} (\log \mathcal{L}) \sum_{p \in \mathcal{F}} 1 \ll N^2 \mathcal{L}^{-3-2\theta_0} (\log \mathcal{L})^4 \tag{38}$$

and then using (37), (38) and Lemma 4 we get

$$\mathcal{E}_2'' \ll N^2 \mathcal{L}^{-3-2\theta_0} (\log \mathcal{L})^4 \sum_{p \leq N} \left| \sum_{\substack{D < d < N/D \\ d|p-1}} \chi(d) \right|^2 \ll N^3 \mathcal{L}^{-4-2\theta_0} (\log \mathcal{L})^{11}. \tag{39}$$

From (35), (36) and (39) we conclude that

$$\mathcal{E}_2 \ll N^2 \mathcal{L}^{-\theta_0} (\log \mathcal{L})^6. \tag{40}$$

3.4 The estimation of \mathcal{E}_3

From (16) it follows that

$$\begin{aligned}
S_3(n) &= \sum_{p_1+p_2=n} \log p_1 \log p_2 \sum_{\substack{m|p_1-1 \\ \frac{p_1-1}{m} \geq \frac{N}{D}}} \chi\left(\frac{p_1-1}{m}\right) \\
&= \sum_{p_1+p_2=n} \log p_1 \log p_2 \sum_{j=\pm 1} \chi(j) \sum_{\substack{m \leq \frac{(p_1-1)D}{N}, \ 2|m \\ p_1 \equiv 1+jm \pmod{4m}}} 1.
\end{aligned}$$

We change the order of summation and use (9) to find

$$S_3(n) = \sum_{\substack{m \leq D \\ 2|m}} \sum_{j=\pm 1} \chi(j) J_{4m, 1+jm}(n, I_m),$$

where I_m denotes the interval $[1 + mN/D, N]$. Having in mind Lemma 1 we write

$$S_3(n) = S_3'(n) + S_3^*(n), \tag{41}$$

where

$$\begin{aligned}
S'_3(n) &= \sum_{\substack{m \leq D \\ 2|m}} \sum_{j=\pm 1} \chi(j) \frac{\mathfrak{S}_{4m,1+jm}(n)}{\varphi(4m)} \Phi(n, I_m), \\
S_3^*(n) &= \sum_{\substack{m \leq D \\ 2|m}} \sum_{j=\pm 1} \chi(j) \left(J_{4m,1+jm}(n, I_m) - \frac{\mathfrak{S}_{4m,1+jm}(n)}{\varphi(4m)} \Phi(n, I_m) \right). \quad (42)
\end{aligned}$$

Since $2 \mid n$ it follows from (10) that

$$\mathfrak{S}_{4m,1+jm}(n) = \begin{cases} c_0 \lambda(4mn) & \text{if } (4m, n-1-jm) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

However the condition $(4m, n-1-jm) = 1$ is independent of j (from the set $\{1, -1\}$) and therefore $\mathfrak{S}_{4m,1+jm}(n)$ is independent of j too. This means that

$$S'_3(n) = 0.$$

Hence, using (12), (18), (41), (42) and Lemma 1 we find

$$\begin{aligned}
\mathcal{E}_3 &\ll \sum_{n \leq N} |S_3^*(n)| \\
&\ll \sum_{\substack{m \leq D \\ 2|m}} \sum_{j=\pm 1} \sum_{n \leq N} \left| J_{4m,1+jm}(n, I_m) - \frac{\mathfrak{S}_{4m,1+jm}(n)}{\varphi(4m)} \Phi(n, I_m) \right| \\
&\ll \sum_{k \leq 4D} \max_{(l,k)=1} \max_{I \in \mathcal{I}} \sum_{n \leq N} \left| J_{k,l}(n, I) - \frac{\mathfrak{S}_{k,l}(n)}{\varphi(k)} \Phi(n, I) \right| \\
&\ll N^2 \mathcal{L}^{-1}. \quad (43)
\end{aligned}$$

The estimate (8) follows from (17), (34), (40) and (43), so the theorem is proved.

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